Example
Suppose we wish to trigger an alarm system whenever a given signal voltage is present. There is also additive noise present; the noise voltage is Gaussian-distributed with zero mean value. The alarm has a preset threshold such that whenever the total voltage exceeds the threshold the alarm triggers and rings a bell.

(a) It is desired to set the threshold at the lowest voltage level possible and yet have the alarm ring the bell no more than 1% of the time, on the average, when there is no signal present. Determine the required threshold voltage if the rms value of the noise is 0.5V.

(b) Assume that when the signal is present, this signal is a positive 2V. With the threshold set as in part (a), what is the probability that the bell will not ring even when the signal is present

Solution:
Let $Y$ be the input to the alarm, $X$ be the signal and $N$ be the noise

Gaussian distribution (pdf): $f(n) = \frac{1}{\sqrt{2\pi}\sigma_N} e^{-(n-\mu_N)^2/2\sigma_N^2}$

(a) No signal $\Rightarrow Y=N$, i.e. $f(y) = f(n)$
Mean value of $Y$, $\mu_Y = \mu_N = 0$ (noise $N$ has zero mean)
Standard deviation of $Y$, $\sigma_Y = \sigma_N = 0.5$ (rms of $N$ is 0.5)

$f(y) = \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-(y-\mu_Y)^2/2\sigma_Y^2}$

Let $y_o$ be the threshold that the alarm will ring when $y > y_o$

$P(\text{ring}) = \int_{y_o}^{\infty} \frac{1}{\sqrt{2\pi}(0.5)} e^{-y^2/2(0.5)^2} dy$

$= \int_{y_o}^{\infty} \frac{2}{\sqrt{2\pi}} e^{-2y^2} dy$

For $P(\text{ring}) = 0.01$,

$\int_{y_o}^{\infty} \frac{2}{\sqrt{2\pi}} e^{-2y^2} dy = 0.01$

Let $a=2y$,

$\int_{2y_o}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-a^2/2} da = 0.01$

$Q(2y_o) = 0.01$

By table, $y_o = 1.163V$

$Q(a) = \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} e^{-s^2/2} ds$

So, the required threshold value is 1.163V
(b) When the signal $X$ is present $\rightarrow Y = X + N$

Signal $X$ is a constant value of $+2V$.  

Mean value of $Y$, $\mu_Y = 2 + \mu_N = 2$ (noise $N$ has zero mean) 

Standard deviation of $Y$, $\sigma_Y = \sigma_N = 0.5$ (rms of $N$ is 0.5)

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}}$$

Use the same threshold, $y_o$, that the alarm will ring when $y > y_o$

$$P(\text{NOT ring}) = P(Y \leq y_o)$$

$$= \int_{-\infty}^{y_o} \frac{1}{\sqrt{2\pi}(0.5)} e^{-\frac{(y-2)^2}{2(0.5)^2}} dy$$

Let $a=2(y-2)$,

$$= \int_{-\infty}^{-1.674} \frac{1}{\sqrt{2\pi}} e^{-\frac{a^2}{2}} da$$

$$= \int_{1.674}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{a^2}{2}} da$$

$$= Q(1.674)$$

$$= 0.0471$$

**Example:**

A certain data transmission system has an average error probability of $1 \times 10^{-6}$ per digit. Determine the probability that in a message of $2 \times 10^6$ digits, there will be

(a) exactly two errors;  
(b) less than two errors;  
(c) more than two errors

**Solution:**

Given: $N=2 \times 10^6$, $P_e=1 \times 10^{-6}$

Mean number of errors, $\lambda = NP_e = (2 \times 10^6) (1 \times 10^{-6}) = 2$

Using Poisson distribution, $P(X = i) = e^{-\lambda} \frac{\lambda^i}{i!}$ where $\overline{X} = \lambda$

(a) $P(2 \text{ errors}) = P(X=2) = e^{-2} \frac{2^2}{2!} = 2e^{-2} = 0.2707$

(b) $P(<2 \text{ errors}) = P(X<2) = P(X=0)+P(X=1)$

$$= e^{-2}(0^0/0!) + e^{-2}(2^1/1!)$$

$$= e^{-2}+ 2e^{-2}$$

$$= 0.406$$

(c) $P(>2 \text{ errors}) = P(X>2) = 1 - P(X=0) - P(X=1) - P(X=2)$

$$= 1- e^{-2}(2^0/0!) - e^{-2}(2^1/1!) - e^{-2}(2^2/2!)$$

$$= 1- e^{-2} - 2e^{-2} - 2e^{-2}$$

$$= 0.3233$$
**Example**

A random variable $X$ has the following pdf:

\[ f(x) = \begin{cases} 
  kx(1-x) & \text{if } 0 \leq x < 1 \\
  0 & \text{elsewhere}
\end{cases} \]

(a) Find the numerical value of $k$.
(b) Find $P(X \leq 1/4)$

**Solution:**

(a) Using \[ \int_{-\infty}^{\infty} f(x) \, dx = 1, \]

\[ \int_{0}^{1} kx(1-x) \, dx = 1 \]

\[ k \left( \frac{x^2}{2} - \frac{x^3}{3} \right) \bigg|_{0}^{1} = 1 \]

\[ k \left( \frac{1}{2} - \frac{1}{3} \right) = 1 \]

\[ k \left( \frac{1}{6} \right) = 1 \]

\[ k = 6 \]

(b)

\[ f(x) = \begin{cases} 
  6x(1-x) & \text{if } 0 \leq x < 1 \\
  0 & \text{elsewhere}
\end{cases} \]

\[ P(X \leq 1/4) = \int_{0}^{1/4} 6x(1-x) \, dx \]

\[ = 6 \left( \frac{x^2}{2} - \frac{x^3}{3} \right) \bigg|_{0}^{1/4} \]

\[ = 0.1563 \]
Example:
Let $X$ be a random variable that takes on the values $\pm A$ with equal probabilities. Let $Y$ be a Gaussian random variable with zero mean and variance $\sigma^2$. Assume that $X$ and $Y$ are statistically independent, and $A > \sigma$.

Given that $Z = X + Y$ has a pdf of

$$f(z) = \frac{0.5}{\sqrt{2\pi\sigma}} e^{-(z+A)^2/(2\sigma^2)} + \frac{0.5}{\sqrt{2\pi\sigma}} e^{-(z-A)^2/(2\sigma^2)}$$

The random variable $Z$ is available to an observer to make an estimate if $X=+A$ or $X=-A$ is present. The rule adopted is that if $Z>0$ then the decision is that $+A$ is present and if $Z<0$ then $-A$ is present. What is the probability of the observer making an error?

Solution:

$$f(z) = \frac{0.5}{\sqrt{2\pi\sigma}} e^{-(z+A)^2/(2\sigma^2)} + \frac{0.5}{\sqrt{2\pi\sigma}} e^{-(z-A)^2/(2\sigma^2)}$$

$$P_e = \int_{-\infty}^{0} \frac{0.5}{\sqrt{2\pi\sigma}} e^{-(z+A)^2/(2\sigma^2)} dz + \int_{-\infty}^{0} \frac{0.5}{\sqrt{2\pi\sigma}} e^{-(z-A)^2/(2\sigma^2)} dz$$

$$= \int_{-\infty}^{A/\sigma} \frac{0.5}{\sqrt{2\pi}} e^{-u^2/\sigma} du + \int_{-\infty}^{A/\sigma} \frac{0.5}{\sqrt{2\pi}} e^{-v^2/\sigma} dv$$

$$= \int_{-\infty}^{A/\sigma} \frac{0.5}{\sqrt{2\pi}} e^{-u^2/\sigma} du + \int_{-\infty}^{A/\sigma} \frac{0.5}{\sqrt{2\pi}} e^{-v^2/\sigma} dv$$

$$Q(a) = \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} e^{-s^2/2} ds$$

Thus the dimensionless ratio $A/\sigma$ completely controls the error probability.