Signal Detection in Optical Communications through the Atmospheric Turbulence Channel

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Abstract—The probability of a miss in the detection of a signal in an optical communications system through the turbulent atmosphere using intensity modulation is studied. The turbulence of the atmosphere causes scintillation of the received signal intensity which is treated as a lognormal random process. The received background radiation and electronic noise in the receiver is treated as additive white Gaussian noise (AWGN). A Chernoff bound is derived. For practical values of the signal-to-noise power ratio (SNR), a series solution for the characteristic function of the lognormal random variable is used to find the probability of miss. Simulation results agree with theoretical results. The method developed in this paper can be used by the system designer to choose the proper signal length and meet the system specifications for signal detection.

I. INTRODUCTION

Optical communications through the atmosphere is important to commercial and defense applications. The advantages of optical communications through the atmosphere over radio communications include increased security and unlicensed, large bandwidth. Some of the applications of optical communications through the atmosphere include last-mile links [1] and mobile battlefield networks [2]. One of the challenges to optical communications through the atmosphere is overcoming the effect of turbulence on the signal as it propagates through the atmosphere. Atmospheric turbulence is caused by small fluctuations in temperature and pressure, which affects the propagating optical signal as random changes in the index of refraction [3]. At the receiver focal plane, this causes small fluctuations (scintillation) of the received optical intensity and polarization, which can degrade the performance of the communication system.

In packet-switched networks, the information is sent out in packets in which the first portion of each packet is a unique sequence of symbols (the preamble), which is known to the receiver. The receiver determines that a packet is present when it detects the presence the preamble. Once the receiver detects that a packet is present, it performs parameter estimation, synchronization and demodulation. The detection of this preamble in an atmospheric turbulence channel is a discrete signal detection problem expressed by the hypothesis test

\[ H_0 : r_k = n_k, \quad k = 1, 2, \ldots, L \]  
versus

\[ H_1 : r_k = A_k d_k + n_k \quad k = 1, 2, \ldots, L, \]  

where \( \mathbf{r} = [r_1, r_2, \ldots, r_L] \) is the observation vector, \( \{n_k\} \) are samples of an AWGN process representing the thermal noise of the receiver, \( L \) is the signal length, \( \mathbf{d} = [d_1, d_2, \ldots, d_L] \) is the sequence to be detected and \( \{A_k\} \) are samples from the stochastic process determined by the turbulent atmospheric channel which controls the strength of the signal. When the signal is present, the observation is the signal degraded by the channel plus the AWGN, whereas only the AWGN is observed when the signal is absent. During the signal detection operation, one of following two events may occur to degrade the performance of the system: 1) miss detection or 2) false alarm. Miss detection is false rejection of \( H_1 \) and false alarm is the false rejection of \( H_0 \). The probability of a miss detection and the probability of a false alarm are important measures in the design and implementation of a wireless communication link. Much of the groundwork for atmospheric communications theory has been laid by Hoversten et al. [4]. In this paper, we focus on calculating the probability of miss detection for optical communications through the turbulent atmosphere.

The rest of this paper is organized as the following. In Section II, the system model is described. In Section III, the Chernoff bound is applied to the lognormal sum. In Section IV, a series approximation to the characteristic function of a lognormal random variable is applied to the problem. In Section V, the detection problem is studied with AWGN included. Conclusions are provided in Section VI.

II. SYSTEM MODEL

Suppose that a binary sequence \( \{d_i\}_{i=1}^l \) of length \( l \) will be transmitted over a free-space channel. To accomplish this, a system is required in which a transmitter prepares the sequence for transmission over the channel and a receiver detects the signal when it arrives. Figure 1 illustrates the model for such a system. The transmitter is composed of an electrical modulator, an optical modulator and a transmitting telescope. In the electrical modulator an information sequence is impressed upon an electrical signal. This electrical signal is then used to modulate the intensity of an optical source. On-off keying (OOK) is widely employed for optical intensity modulation in optical communications through the atmosphere [3]. After the signal propagates through the atmospheric channel, it passes through the receiver optics and enters the photodetector, which converts the optical signal intensity into an electrical signal.
The electrical signal is then sent to the detector, in which it is determined whether the signal has arrived. When a signal is detected, the detector triggers the demodulator.

OOK is a binary modulation scheme in which the optical source is switched on to transmit bit "1" and turned off to transmit bit "0". The transmitted optical signal is the random process

\[ s(u, t) = I_s \sum_{i=1}^{l} d_i g(t - iT_u), \]  

where \( I_s \) is the average intensity of the field, \( \{d_i\}_{i=1}^{l} \) (with \( d_i \in \{0, +1\} \)) is a unique sequence, \( g(t) \) is the shaping pulse and \( T_u \) is the symbol time [3]. At the photodetector, the received optical intensity is

\[ x(u, t) = A(u, t)K \sum_{i=1}^{l} d_i g(t - iT_u) + n(u, t), \]  

where \( K \) is a constant determined by the received intensity and the photoelectric conversion efficiency of the detector and \( n(u, t) \) is an AWGN process. Without loss of generality, the constant \( K \) can be dropped. After \( x(u, t) \) is sampled at time \( t = iT_u \), each bit is given its corresponding random variable:

\[ r_i(u) = \begin{cases} n_i(u), & \text{(bit "0") } \\
A_i(u) + n_i(u), & \text{(bit "1") } 
\end{cases} \]  

Thus, when the full signal for detection is in the correlator, the bit "0" random variables drop out because they are multiplied by 0 and the correlator output is

\[ c(u) = \sum_{i=1}^{W} A_i(u) + n_i(u), \]  

where \( W < l \) is the weight of the signal sequence.

In (2), the stochastic process \( A \) is the representation of scintillation of the signal \( S \). When the strength of the scintillation is small, this process can be modeled as a lognormal random process [5] given by

\[ A(u, t) = e^{x(u, t)}, \]  

where \( x(u, t) \) is a stationary Gaussian random process with mean \( \mu \) and variance \( \sigma^2 \). The probability density function (PDF) of \( A \) is

\[ f(x) = \frac{1}{\sqrt{2\pi \sigma^2}} \exp \left\{ \frac{(\ln x + \mu)^2}{2\sigma^2} \right\}, \]  

where the variance \( \sigma^2 \) is a measure of the strength of the scintillation. In this paper, \( \mu \) is set to 0. The value of \( \sigma \) is often referred to as the scintillation index. In [6], scintillation measurements were performed on a typical day and \( \sigma \) never increased beyond about 0.75. In this paper, the probability of a miss detection will be found for the range of scintillation index appropriate for near-earth horizontal laser propagation.

The signal detector declares the arrival of a signal if

\[ c(u) \geq \alpha, \]  

where \( \alpha \) is a predetermined threshold. For OOK systems, this means computing the sum of \( W \) lognormal and \( W \) Gaussian random variables. In this paper, it is assumed that the correlation time of the atmospheric turbulence is less than the symbol time \( T_u \); therefore, the lognormal random variables are independent. The determination of \( \alpha \) is critical in system design [7]. If the detection sequence is present but \( c(u) < \alpha \), the signal was missed. If the detection sequence was not sent but \( c(u) \geq \alpha \), a false alarm has occurred.

Finding the probability of miss means finding the cumulative distribution function (CDF) of \( c(u) \). If the SNR is taken to be \( \infty \), \( c(u) \) is the sum of \( L \) i.i.d. lognormal random variables. In the next two sections, different methods to find the CDF of the sum of multiple i.i.d. lognormal random variables will be explored.

### III. CHERNOFF BOUND ON LOGNORMAL SUM DISTRIBUTION

In this section, a Chernoff bound is derived to find a bound on the distribution of a sum of \( L \) lognormal random variables. Suppose that \( \{X_i\}_{i=1}^{L} \) is a set of lognormal random variables, each with mean \( m \neq 0 \). The Chernoff bound for the sum of these random variables is

\[ \mathbb{E} \left\{ e^{\lambda_0 (e^X - (m + \epsilon))} \right\} \geq \begin{cases} P \left\{ \frac{1}{L} \sum_{i=1}^{L} e^{X_i} \geq m + \epsilon \right\}, & \epsilon > 0 \\
P \left\{ \frac{1}{L} \sum_{i=1}^{L} e^{X_i} \leq m + \epsilon \right\}, & \epsilon < 0 \end{cases} \]  

where \( \lambda_0 \) is defined by

\[ \mathbb{E} \left\{ e^{\lambda_0 X} \right\} = m + \epsilon. \]
The fact that
\[ \mathbf{E}\{e^{tX}\} = \infty, \forall t > 0 \] (13)
makes it difficult to find \( \lambda_0 \) in (12). Therefore, an approximation for \( e^X \) will be made using the Taylor series
\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \] (14)

The following three integrals are used in computing the Chernoff bound using the third order Taylor approximation of \( e^x \):
\[ I_1 = \mathbf{E}\{e^{\lambda_0(x+x^2/2)}\} = \frac{1}{\sqrt{1-\sigma^2\lambda_0}} \exp\left\{\frac{\sigma^2\lambda_0^2}{2(1-\sigma^2\lambda_0)}\right\}, \] (15)
\[ I_2 = \mathbf{E}\{xe^{\lambda_0(x+x^2/2)}\} = \frac{\sigma^2\lambda_0}{(1-\sigma^2\lambda_0)^{3/2}} \exp\left\{\frac{\sigma^2\lambda_0^2}{2(1-\sigma^2\lambda_0)}\right\}, \] (16)
\[ I_3 = \mathbf{E}\{x^2e^{\lambda_0(x+x^2/2)}\} = \frac{\sigma^2}{(1-\sigma^2\lambda_0)^{3/2}} \left(1 + \frac{\sigma^2\lambda_0^2}{1-\sigma^2\lambda_0}\right) \exp\left\{\frac{\sigma^2\lambda_0^2}{2(1-\sigma^2\lambda_0)}\right\}. \] (17)

Now, the equation for \( \lambda_0 \) is
\[ m + \epsilon = \frac{I_1 + I_2 + \frac{1}{2}I_3}{I_1}, \] (18)
which becomes
\[ m + \epsilon = \frac{\alpha}{L} \]
\[ = 1 + \frac{\sigma^2\lambda_0}{(1-\sigma^2\lambda_0)^{3/2}} + \frac{\sigma^2}{2(1-\sigma^2\lambda_0)} \left(1 + \frac{\sigma^2\lambda_0^2}{1-\sigma^2\lambda_0}\right). \] (19)

Equation (19) is plotted in Fig. 2. The graph shows that many of the values of the normalized threshold \( \alpha_N = \alpha/L \) that are important in finding the bound are absent from the range of \( \alpha_N(\lambda_0) \). To find the bound for a certain threshold, first \( \lambda_0 \) is found numerically in (19). Then, \( \lambda_0 \) is plugged into
\[ P\left\{\sum_{i=1}^{L} e^{X_i} \leq \alpha\right\} \leq \left[\frac{1}{\sqrt{1-\sigma^2\lambda_0}} \exp\left\{\frac{\sigma^2\lambda_0^2}{2(1-\sigma^2\lambda_0)} + 1 - \frac{\alpha}{L}\right\}\right]^L. \] (20)

For those values of the threshold which \( \lambda_0 \) can be found the Chernoff bound computed using the third order approximation is tighter than the bound computed using the second order approximation. No Chernoff bound is present in Fig. 2 for \( \sigma = 1.0 \) because no values of \( \lambda_0 \) exist when \( \alpha_N < 1 \) as shown in Fig. 2. The computation of the integrals in (15)-(17) was simple, but if the fourth term in the Taylor series for \( e^x \) was added these integrals become difficult to compute.

In this section, we have seen that the computation of the Chernoff bound using the third order Taylor approximation for \( e^x \) gives a rough upper bound for the lognormal sum CDF for small values of \( \sigma \).

IV. CALCULATION OF LOGNORMAL SUM DISTRIBUTION USING LOGNORMAL CHARACTERISTIC FUNCTION

One method that can be used to find the CDF of a sum of random variables involves using the characteristic functions of the random variables in the sum. The characteristic function
of a random variable $X$ is defined as

$$\phi_x(\tau) = \int_{-\infty}^{\infty} f_x(x)e^{j\tau x} dx,$$

where $f_x(x)$ is the PDF of $X$. The characteristic function of the sum of multiple random variables is the product of the characteristic functions of the random variables in the sum; therefore,

$$\phi_y(\tau) = \prod_{k=1}^{L} \phi_{x_k}(\tau),$$

where $\phi_{y}(\tau)$ is the characteristic function of the lognormal sum and $\phi_{x_k}$ is the characteristic function of the $k$th lognormal random variable. The PDF of $Y$ is the inverse Fourier transform of $\phi_y(\tau)$ given by

$$f_y(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_y(\tau)e^{-j\tau y} d\tau.$$  \hspace{1cm} (23)

The CDF of $Y$ can be found by integrating the PDF as in

$$F_y(y) = \int_{-\infty}^{y} f_y(u) du.$$  \hspace{1cm} (24)

Thus, with knowledge of the characteristic function of a lognormal random variable, the CDF of a lognormal sum can be calculated.

In [8], the following series is given for the characteristic function of a lognormal random variable:

$$\phi_x(\tau) = e^{j\tau - \tau^2\sigma^2/2} \sum_{n=0}^{\infty} \left(\frac{\tau^2}{n!}\right) a_n(j\tau) h_n(\sigma \tau),$$

where $a_n$ are Taylor series coefficients given by

$$a_n(j\tau) = \frac{d^n}{dz^n} \exp\{j\tau (e^z - z - 1)\} \big|_{z=0}$$

and $h_n$ are Hermite polynomials given by the recursive formula

$$h_0(z) = 1,$n \geq 1, h_1(z) = z, h_{n+1}(z) = z h_n(z) - nh_{n-1}(z).$$

The first twelve coefficients $a_n$ and first six hermite polynomials $h_n$ are given in [8] and can be obtained with the aid of a symbol-manipulating program such as Maple. This function converges quickly for $\sigma < 1$, which is the range of $\sigma$ applicable to atmospheric turbulence in optical communications. Although less terms may have sufficed, twelve terms were used for the sum in this paper.

Computation of the PDF is difficult because of the oscillatory nature of $\phi(\tau)$. In practice, the integration limit $T$ in

$$f_y(y) = \frac{1}{2\pi} \int_{-T}^{T} \phi_y(\tau)e^{-j\tau y} d\tau.$$  \hspace{1cm} (28)

will have to be set at a carefully selected finite value. If $T$ is too small, too much of the integration will be shut out and if it is too large, the computation of the characteristic function in (25) near the limits may blow up due to numerical inaccuracies. The CDF of $Y$ is found by integrating (28). Figure 4 shows the CDF for four different values of $\sigma$. Comparison of the calculated CDF with simulations shows that this method produces accurate results.

V. PROBABILITY OF MISS

A communications system through the atmosphere is power-limited; therefore, the effect of the AWGN inherent in electrical systems must be included. In this section, the probability of miss will be calculated for practical values of SNR in optical communications through the atmosphere.

Noting that (22) is valid even if the characteristic functions in the product are different, the AWGN can be included in the calculation of the probability of miss by including the Gaussian characteristic function. The characteristic function of $c(u)$ is, thus, given by

$$\phi_c(\tau) = \prod_{k=1}^{L} \phi_{x_k}(\tau)\phi_{g_k}(\tau),$$

where $\phi_{g}(\tau)$ is the characteristic function of a zero-mean Gaussian random variable given by

$$\phi_{g}(\tau) = \exp\left\{-\frac{1}{2\sigma_g^2}\tau^2\right\}$$

and $\sigma_g^2$ is the power of the AWGN.

The CDF of $c(u)$ for a signal of length $L = 108$ and two values of the scintillation index $\sigma$ is shown in Fig. 5 for SNR = 30 and 10 dB. The solid curves are the calculated probability of miss and the simulation points are represented by the indicated shapes for each scintillation index. The theoretical curves match well with simulation for low to moderate strengths of atmospheric turbulence. Figure 5 shows the diminishing effect of the scintillation strength on the probability of miss as the SNR decreases.
A good choice for the threshold setting is the point at which the probability of false alarm is equal to the probability of miss. For this system, the probability of false alarm is the probability that the sum of \( L \) zero-mean Gaussian random variables is greater than \( \alpha \) which is

\[
P_{FA}(\alpha) = \frac{1}{2} \left[ 1 - \text{erf} \left( \frac{\alpha}{\sqrt{2L\sigma^2}} \right) \right]. \tag{31}
\]

For an optical wireless system operating through the turbulent atmosphere, we look at the worse case scenario for probability of miss, which is when \( \sigma \) is maximum. For \( L = 108 \), the probability of miss and the probability of false alarm do not intersect at any practical value for the probability of miss; hence, a shorter signal length can be used.

For a signal of length \( L = 32 \), the false alarm rate and miss rate are plotted in Fig. 6 and Fig. 7 for \( \text{SNR} = 10 \text{ dB} \) and \( \text{SNR} = 5 \text{ dB} \), respectively. Figure 6 shows that a system operating at \( \text{SNR} = 10 \text{ dB} \) could use a signal of length 32 with a detection threshold set at \( \alpha = 12 \) in order to obtain a probability of false alarm or miss detection that is below \( 10^{-10} \). When the \( \text{SNR} = 5 \text{ dB} \), Fig. 7 shows that the same performance can not be achieved with high scintillation.

**VI. CONCLUSIONS**

In this paper, we have shown how to find the probability of miss for an optical system operating through the atmosphere in the presence of lognormal fading and AWGN. For low to moderate scintillation strengths the probability of miss can be found by numerical integration. For a signal length \( L = 32 \), the probability of miss has been plotted for \( \text{SNR} = 5 \text{ and 10 dB} \). By plotting the probability of miss, the system designer can find the appropriate parameters to meet performance requirements.

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**REFERENCES**


